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Electromagnetic wave propagation in a liquid crystal with slowly varying wavelength

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Abstract. The theory of electromagnetic waves in a liquid crystal with the dielectric tensor varying from plane to plane is developed, using the method of geometrical optics, or equivalently the WKB approximation. The approximation is shown to apply when the wavelength is slowly varying and when the wavelength is small compared with the pitch of the director. Only normal propagation is considered. Explicit formulae for the propagating fields are derived. It is found that in first approximation the fields are plane polarised in directions that rotate around the normal to the planes in step with the director.

1. Introduction

There are only a few cases in which an exact solution of Maxwell's equations for propagation of electromagnetic waves in a liquid crystal can be made. Mauguin [1], Oseen [2], and de Vries [3] gave the analysis for a cholesteric structure, in which the director has the form $\boldsymbol{n}(z) = (\cos\phi, \sin\phi, 0)$, with ϕ proportional to z. Their work was reviewed and extended by de Gennes [4], by Peterson [5] and by Oldano *et al* [6]. Ong and Meyer [7] gave the solution in case of a periodically bent nematic crystal, having $\boldsymbol{n}(z) = (\sin\theta, 0, \cos\theta)$, with θ proportional to z. (Also, they gave a very valuable list of references to previous work and applications.) However, for more complicated variation of the director in space, since there is no exact treatment, there is motivation for an approximate discussion that shows semiquantitatively what can take place.

In this paper it is shown that a complete treatment of the wave propagation in a liquid crystal, with the dielectric tensor varying in one direction, can be made in a geometrical optics or WKB type of approximation. That is, the dielectric constant of the form

$$\varepsilon_{\alpha\beta}(z) = \varepsilon_{\perp}\delta_{\alpha\beta} + (\varepsilon_{\parallel} - \varepsilon_{\perp})n_{\alpha}n_{\beta}$$

with arbitrary dependence on $\dot{z}$ is considered. The director may have any direction, so $n(z) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, and θ and ϕ are not necessarily linear in z. Thus, cholesterics with non-helical directors and smectics with variation of $\varepsilon_{\perp}$, $\varepsilon_{\parallel}$, $\boldsymbol{n}$ from plane to plane are treated.

It turns out that the approximation applies only when the wavelength is slowly varying ($\mathrm{d}\lambda/\mathrm{d}z < 1$) and when it is small compared with the pitch of the director ($(c/\omega)\,\mathrm{d}\phi/\mathrm{d}z < 1$) so the approximation holds only when the variation of $\varepsilon_{\alpha\beta}$ with z over a wavelength is small.

There are two modes of propagation, say A and B, which can be described conveniently in terms of unit vectors $\boldsymbol{k}$ and $\boldsymbol{r}$, where $\boldsymbol{k}$ is in the z direction and $\boldsymbol{r}$ is in the direction of the projection of the director $\boldsymbol{n}$ in the xy plane. It is shown that, in a first approximation, mode A has in-phase components of $\boldsymbol{E}$ in the $\boldsymbol{r}$ and $\boldsymbol{k}$ directions, mode B in the $\boldsymbol{k} \times \boldsymbol{r}$ direction only. In the next approximation, out-of-phase components come in in the other directions. Explicit formulae for these components are obtained.

Previously Ong and Meyer [8] applied the geometrical optics approximation in the case where $\boldsymbol{n}(z) = (\sin\theta, 0, \cos\theta)$, with $\theta(z)$ arbitrary. The approximation itself, as applied to Maxwell fields in a material with a dielectric tensor, was reviewed by Nave and Gibbons [9]. The geometrical optics approximation, for a system with $\varepsilon_{\alpha\beta}$ a function of z only, coincides with the WKB approximation, as generalised to a many-component problem. One makes the generalisation, as in § 3 below, using ideas developed by Pauli [10] in a quantum mechanical application.

Another important technique, which can be applied to the study of transmission in a material with a z-dependent dielectric tensor, is the transfer matrix method. This was reviewed and applied by, for example, Sprokel [11] and by Khoo and Hou [12]. The present method gives a quick physical picture of what happens for normal propagation but the transfer matrix method applies more generally for non-normal propagation.

2. Basic equations

The material considered is non-magnetic, so $\boldsymbol{B} = \mu_0\boldsymbol{H}$, but has a tensor dielectric constant, so $D_\alpha = \varepsilon_0\varepsilon_{\alpha\beta}E_\beta$, with $\varepsilon_{\alpha\beta}$ varying in one direction only, say z. The starting point is Maxwell's equations:

$$\begin{aligned} &\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t \qquad &&\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t \\ &\nabla \cdot \boldsymbol{B} = 0 &&\nabla \cdot \boldsymbol{D} = 0 \end{aligned} \tag{1}$$

in SI units. In the usual way, complex solutions will be found and the real parts may be taken at the end of the calculation.

The time dependence $\exp(-\mathrm{i}\omega t)$ is assumed, in which case the divergence equations follow from the curl equations. Then if $\boldsymbol{H}$ is eliminated, the problem of finding solutions reduces to the problem of solving the equation

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = \mu_0 \omega^2 \boldsymbol{D}.$$

Axial propagation is considered so there is no dependence on x and y. In this case the problem becomes

$$-\partial^2 E_x/\partial z^2 = (\omega/c)^2(\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z) \tag{2a}$$

$$-\partial^2 E_y/\partial z^2 = (\omega/c)^2(\varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z) \tag{2b}$$

$$0 = (\omega/c)^2(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z). \tag{2c}$$

Here one can solve for E_z

$$E_z = -(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y)/\varepsilon_{zz} \tag{3}$$

and substitute back to get to the two-by-two system

$$(\partial^2/\partial z^2 + \zeta_{xx}/h^2)E_x + (\zeta_{xy}/h^2)E_y = 0 \tag{4a}$$

$$(\zeta_{yx}/h^2)E_x + (\partial^2/\partial z^2 + \zeta_{yy}/h^2)E_y = 0 \tag{4b}$$

where the functions $\zeta_{ij}(z)$ are defined by

$$\zeta_{xx}/h^2 = [(\varepsilon_{xx}\varepsilon_{zz} - \varepsilon_{xz}^2)/\varepsilon_{zz}](\omega/c)^2 \tag{5a}$$

$$\zeta_{xy}/h^2 = \zeta_{yx}/h^2 = [(\varepsilon_{xy}\varepsilon_{zz} - \varepsilon_{xz}\varepsilon_{yz})/\varepsilon_{zz}](\omega/c)^2 \tag{5b}$$

$$\zeta_{yy}/h^2 = [(\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{yz}^2)/\varepsilon_{zz}](\omega/c)^2. \tag{5c}$$

The parameter h is included to mark the order of the terms in the approximation to be made. It cancels out of the calculation later when the results are expressed in terms of $\varepsilon_{\alpha\beta}$.

3. The slowly varying wavelength approximation

A solution is set up as a series in increasing powers of h and the approximation is made by truncating the series. The substitution

$$E_x = (E_x^{(0)} + hE_x^{(1)} + \ldots)\exp(\mathrm{i}/h)\int p\,\mathrm{d}z \tag{6a}$$

$$E_y = (E_y^{(0)} + hE_y^{(1)} + \ldots)\exp(\mathrm{i}/h)\int p\,\mathrm{d}z \tag{6b}$$

is made into (4) and the solution is considered in different orders of h. Here $E_{x,y}^{(j)}$, p, and $\int p\,\mathrm{d}z$ are all functions of z, to be determined. Physically one understands that $\boldsymbol{E}^{(0)} + h\boldsymbol{E}^{(1)}$ determines the polarisation directions of the propagating modes and that $2\pi h/p$ is the local wavelength λ. The expansion parameter h, standing alone, cannot be given any physical significance since it cancels out at a later stage when the unknowns are determined. Evidently ω/c could be used as the expansion parameter and h not introduced at all, but the present technique allows for solving (4) when ζ is not proportional to $(\omega/c)^2$. The first contribution to the solution, given by the terms proportional to h^{-2}, is to be found from

$$(-p^2 + \zeta_{xx})E_x^{(0)} + \zeta_{xy}E_y^{(0)} = 0 \tag{7a}$$

$$\zeta_{yx}E_x^{(0)} + (-p^2 + \zeta_{yy})E_y^{(0)} = 0. \tag{7b}$$

The next order is provided by the terms proportional to h^{-1}:

$$(-p^2 + \zeta_{xx})E_x^{(1)} + \zeta_{xy}E_y^{(1)} = -\mathrm{i}(\partial p/\partial z)E_x^{(0)} - 2\mathrm{i}p\,\partial E_x^{(0)}/\partial z \tag{8a}$$

$$\zeta_{yx}E_x^{(1)} + (-p^2 + \zeta_{yy})E_y^{(1)} = -\mathrm{i}(\partial p/\partial z)E_y^{(0)} - 2\mathrm{i}p\,\partial E_y^{(0)}/\partial z. \tag{8b}$$

Evidently (7) just expresses the eigenvalue problem for the two-by-two matrix ζ. Let the solutions of the eigenvalue problem be

$$p_{\mathrm{A}}^2 \qquad C_{\mathrm{A}}\begin{pmatrix} m_{\mathrm{A}x} \\ m_{\mathrm{A}y} \end{pmatrix}$$

and

$$p_{\mathrm{B}}^2 \qquad C_{\mathrm{B}}\begin{pmatrix} m_{\mathrm{B}x} \\ m_{\mathrm{B}y} \end{pmatrix}$$

where $m_x^2 + m_y^2 = 1$ in each case. The eigenvalue problem is to be solved at each value of z so p^2, C, m all are functions of z.

However, the z dependence of C is determined by the requirement that (8) be solvable for the next order $E_x^{(1)}$ and $E_y^{(1)}$. Consider either solution A or B of (7). Then the determinant of the coefficients on the left in (8) is zero. In this case, as is well known, there is a solution only if the solution of the transposed homogenous set of equations is orthogonal to the vector on the right. Since $\zeta_{xy} = \zeta_{yx}$, the solution is m_x, m_y. In the vector on the right $E^{(0)} = Cm$ is to be used. The condition for solvability is then

$$m_x[(\partial p/\partial z)Cm_x + 2p\,\partial(Cm_x)/\partial z] + m_y[(\partial p/\partial z)Cm_y + 2p\,\partial(Cm_y)/\partial z] = 0.$$

Because $m_x^2 + m_y^2 = 1$ this simplifies to

$$(\partial p/\partial z)C + 2p\,\partial C/\partial z = 0$$

which implies that C is proportional to $p^{-1/2}$.

At this stage, with only the $E^{(0)}$-terms in the series retained, the fields are given by

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{1}{\sqrt{p}}\begin{pmatrix} m_x \\ m_y \end{pmatrix} \exp\left(\frac{\mathrm{i}}{h}\int p\,\mathrm{d}z - \mathrm{i}\omega t\right). \tag{9}$$

Although this result gives a useful first impression of the modes of propagation, it is interesting also to look at the next higher level of approximation, as given by the $E^{(1)}$-terms.

To obtain the next order terms in the solution, one must solve (8) for $E_x^{(1)}$ and $E_y^{(1)}$. For solution A for example the equations are

$$(-p_{\mathrm{A}}^2 + \zeta_{xx})E_{\mathrm{A}x}^{(1)} + \zeta_{xy}E_{\mathrm{A}y}^{(1)} = -\mathrm{i}(\partial p_{\mathrm{A}}/\partial z)m_{\mathrm{A}x}/\sqrt{p_{\mathrm{A}}} - 2\mathrm{i}p_{\mathrm{A}}(\partial/\partial z)m_{\mathrm{A}x}/\sqrt{p_{\mathrm{A}}} \tag{10a}$$

$$\zeta_{yx}E_{\mathrm{A}x}^{(1)} + (-p_{\mathrm{A}}^2 + \zeta_{yy})E_{\mathrm{A}y}^{(1)} = -\mathrm{i}(\partial p_{\mathrm{A}}/\partial z)m_{\mathrm{A}y}/\sqrt{p_{\mathrm{A}}} - 2\mathrm{i}p_{\mathrm{A}}(\partial/\partial z)m_{\mathrm{A}y}/\sqrt{p_{\mathrm{A}}}. \tag{10b}$$

Let the solution be

$$E_{\mathrm{A}}^{(1)} = \alpha m_{\mathrm{A}} + \beta m_{\mathrm{B}} \tag{11}$$

where α and β are to be determined. When this form is used in the equations above, the matrix ζ just makes factors of p_{A}^2 and p_{B}^2, so the system simplifies to

$$(-p_{\mathrm{A}}^2 + p_{\mathrm{B}}^2)\beta m_{\mathrm{B}x} = -\mathrm{i}(\partial p_{\mathrm{A}}/\partial z)m_{\mathrm{A}x}/\sqrt{p_{\mathrm{A}}} - 2\mathrm{i}p_{\mathrm{A}}(\partial/\partial z)m_{\mathrm{A}x}/\sqrt{p_{\mathrm{A}}} \tag{12a}$$

$$(-p_{\mathrm{A}}^2 + p_{\mathrm{B}}^2)\beta m_{\mathrm{B}y} = -\mathrm{i}(\partial p_{\mathrm{A}}/\partial z)m_{\mathrm{A}y}/\sqrt{p_{\mathrm{A}}} - 2\mathrm{i}p_{\mathrm{A}}(\partial/\partial z)m_{\mathrm{A}y}/\sqrt{p_{\mathrm{A}}}. \tag{12b}$$

Now β can be found by multiplying by $m_{\mathrm{B}x}$, $m_{\mathrm{B}y}$ and adding. Since m_{A} and m_{B} are orthogonal, the result simplifies to

$$\beta = [-2\mathrm{i}\sqrt{p_{\mathrm{A}}}/(-p_{\mathrm{A}}^2 + p_{\mathrm{B}}^2)](m_{\mathrm{B}x}\,\partial m_{\mathrm{A}x}/\partial z + m_{\mathrm{B}y}\,\partial m_{\mathrm{A}y}/\partial z). \tag{13}$$

Equations (10) do not provide any information on α. Actually α will be determined by the requirement that the next order equations, for $\boldsymbol{E}^{(2)}$, be solvable. However the α-term will be disregarded entirely at this point because it is in the m_A-direction, as is the $\boldsymbol{E}^{(0)}$-term, and so makes a less-interesting contribution. The magnitude of the α-term is discussed in the conclusion. Thus, to order h, one solution is taken to be

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{1}{\sqrt{p_A}}\left[\begin{pmatrix} m_{Ax} \\ m_{Ay} \end{pmatrix} - \frac{2\mathrm{i}p_A h}{-p_A^2 + p_B^2}\left(m_{Bx}\frac{\partial m_{Ax}}{\partial z} + m_{By}\frac{\partial m_{Ay}}{\partial z}\right)\right.$$
$$\left.\times \begin{pmatrix} m_{Bx} \\ m_{By} \end{pmatrix}\right] \exp\left(\frac{\mathrm{i}}{h}\int p_A \,\mathrm{d}z - \mathrm{i}\omega t\right) \tag{14}$$

and the other is found by interchanging subscripts A and B. One finds the two possible modes of propagation by solving the eigenvalue problem for the matrix ζ. There is an overall constant factor allowed in each solution corresponding to the choice of the constant in the integral.

At this point one can see why the expansion on h is for slowly varying wavelength. Disregard the overall constant factor. In (6) the dominant behaviour for small h is provided by the $\exp(\mathrm{i}/h)\int p\,\mathrm{d}z$ factor. The next contribution is provided by the $E^{(0)}$-terms whose size is estimated by $p^{-1/2}$. The $E^{(1)}$-terms give the next higher-order contributions in h and so on. However, one can write, disregarding a constant factor,

$$\frac{1}{\sqrt{p}} = \exp\left(-\frac{1}{2}\int \mathrm{d}z\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right)$$

which leads to

$$\frac{1}{\sqrt{p}}\exp\frac{\mathrm{i}}{h}\int p\,\mathrm{d}z = \exp\frac{\mathrm{i}}{h}\int p\,\mathrm{d}z\left(1 + \frac{\mathrm{i}}{2}\frac{h}{p^2}\frac{\mathrm{d}p}{\mathrm{d}z}\right). \tag{15}$$

The approximation will have validity therefore if

$$\left|\frac{h}{2p^2}\frac{\mathrm{d}p}{\mathrm{d}z}\right| < 1.$$

The local wavelength is

$$\lambda = 2\pi h/p$$

so the criterion for the applicability of the approximation is

$$|\mathrm{d}\lambda/\mathrm{d}z| < 4\pi. \tag{16}$$

An estimate of $\mathrm{d}\lambda/\mathrm{d}z$ is (difference in length of two neighbouring waves)/(wavelength).

4. The dependence on the dielectric constant

The material considered here has dielectric tensor

$$\varepsilon_{\alpha\beta}(z) = \varepsilon_\perp\delta_{\alpha\beta} + (\varepsilon_\parallel - \varepsilon_\perp)n_\alpha n_\beta$$

where $\varepsilon_\perp(z)$ and $\varepsilon_\parallel(z)$ are eigenvalues and the director $\boldsymbol{n}(z)$ has unit length. The eigenvalue $\varepsilon_\parallel$ is associated with eigenvector $\boldsymbol{n}$; the eigenvalue $\varepsilon_\perp$ has as eigenvector any vector perpendicular to $\boldsymbol{n}$. All the results can be expressed in terms of $\varepsilon_\perp$, $\varepsilon_\parallel$, and the polar angles θ, ϕ of $\boldsymbol{n}$:

$$n_x = \sin\theta\cos\phi \qquad n_y = \sin\theta\sin\phi \qquad n_z = \cos\theta.$$

The matrix ζ is found straightforwardly by substituting into the definition (equations (5)). Here and below, h is set equal to unity since it cancels out of the final results anyway. The result is

$$\zeta = \left(\frac{\omega}{c}\right)^2 \left[\varepsilon_\perp + \frac{\varepsilon_\perp(\varepsilon_\parallel - \varepsilon_\perp)\sin^2\theta}{\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta} \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}\right]. \tag{17}$$

The solutions of the eigenvalue problem for this matrix are given in table 1.

Table 1.

Solution	Eigenvalue, p^2	Eigenvector, m
A	$\left(\frac{\omega}{c}\right)^2 \frac{\varepsilon_\perp \varepsilon_\parallel}{\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta}$	$\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$
B	$\left(\frac{\omega}{c}\right)^2 \varepsilon_\perp$	$\begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}$

5. Results

The components E_x and E_y are found by using the solutions of the eigenvalue problem in (9) or (14). The results are conveniently written in terms of unit vectors $\boldsymbol{k}$ and $\boldsymbol{r}$, where $\boldsymbol{k}$ is in the z direction and $\boldsymbol{r}$ is in the direction of the projection of $\boldsymbol{n}$ in the XY plane. In terms of components $\boldsymbol{k}$ is (0, 0, 1) and $\boldsymbol{r}$ is ($\cos\phi$, $\sin\phi$, 0). From E_x and E_y one obtains E_z using (3).

Consider first the approximation carried as far as the $E^{(0)}$-terms only, as given by (9). $\boldsymbol{B}$ is known as $(-\mathrm{i}/\omega)\boldsymbol{\nabla} \times \boldsymbol{E}$. Here only $\boldsymbol{\nabla}_z$ contributes and, in applying it, the dependence on z in the $p^{-1/2}$-factor is disregarded as contributing in lower order in the slowly varying wavelength approximation. Thus $\boldsymbol{B}$ is given simply by $(p/\omega)\boldsymbol{k} \times \boldsymbol{E}$.

For solution A the results are

$$\boldsymbol{E} = \left(\frac{\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta}{\varepsilon_\perp \varepsilon_\parallel}\right)^{1/4} \left(\boldsymbol{r} - \frac{(\varepsilon_\parallel - \varepsilon_\perp)\sin\theta\cos\theta}{\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta}\boldsymbol{k}\right)$$
$$\times \exp \mathrm{i}\left[\frac{\omega}{c}\int \left(\frac{\varepsilon_\perp \varepsilon_\parallel}{\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta}\right)^{1/2} \mathrm{d}z - \omega t\right] \tag{18a}$$

$$\boldsymbol{B} = \frac{1}{c}\left(\frac{\varepsilon_\perp \varepsilon_\parallel}{\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta}\right)^{1/4} \boldsymbol{k} \times \boldsymbol{r}$$
$$\times \exp \mathrm{i}\left[\frac{\omega}{c}\int \left(\frac{\varepsilon_\perp \varepsilon_\parallel}{\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta}\right)^{1/2} \mathrm{d}z - \omega t\right] \tag{18b}$$

and for solution B one finds

$$\boldsymbol{E} = \varepsilon_\perp^{-1/4}\boldsymbol{k}\times\boldsymbol{r}\exp\mathrm{i}\left(\frac{\omega}{c}\int\varepsilon_\perp^{1/2}\,\mathrm{d}z - \omega t\right) \tag{19a}$$

$$\boldsymbol{B} = -\frac{1}{c}\varepsilon_\perp^{1/4}\boldsymbol{r}\exp\mathrm{i}\left(\frac{\omega}{c}\int\varepsilon_\perp^{1/2}\,\mathrm{d}z - \omega t\right). \tag{19b}$$

Positive roots are used in the integrands so as to have propagation in the positive z direction.

It is seen that, in mode A, $\boldsymbol{E}$ has a longitudinal component but otherwise all the fields are transverse. In mode A, $\boldsymbol{E}$ is partly in the $\boldsymbol{r}$ direction; in mode B, $\boldsymbol{B}$ is in the $\boldsymbol{r}$ direction. All the fields have a polarisation direction that rotates about the z axis in step with the director.

The next approximation, which includes the $E^{(1)}$-terms, is provided by using the solutions of the eigenvalue problem in (14). For solution A the result is

$$\begin{aligned}\boldsymbol{E} = {}& \left(\frac{\varepsilon_\perp\sin^2\theta + \varepsilon_\parallel\cos^2\theta}{\varepsilon_\perp\varepsilon_\parallel}\right)^{1/4}\Bigg[\boldsymbol{r} - 2\mathrm{i}\frac{\varepsilon_\parallel}{(\varepsilon_\perp - \varepsilon_\parallel)\sin^2\theta} \\ &\times\left(\frac{\varepsilon_\perp\sin^2\theta + \varepsilon_\parallel\cos^2\theta}{\varepsilon_\perp\varepsilon_\parallel}\right)^{1/2}\frac{c}{\omega}\frac{\mathrm{d}\phi}{\mathrm{d}z}\boldsymbol{k}\times\boldsymbol{r} - \frac{(\varepsilon_\parallel - \varepsilon_\perp)\sin\theta\cos\theta}{\varepsilon_\perp\sin^2\theta + \varepsilon_\parallel\cos^2\theta}\boldsymbol{k}\Bigg] \\ &\times\exp\mathrm{i}\left[\frac{\omega}{c}\int\left(\frac{\varepsilon_\perp\varepsilon_\parallel}{\varepsilon_\perp\sin^2\theta + \varepsilon_\parallel\cos^2\theta}\right)^{1/2}\mathrm{d}z - \omega t\right]\end{aligned} \tag{20}$$

and for solution B one finds

$$\begin{aligned}\boldsymbol{E} = {}& \varepsilon_\perp^{-1/4}\Bigg[\boldsymbol{k}\times\boldsymbol{r} - 2\mathrm{i}\frac{\varepsilon_\perp^{1/2}\varepsilon_\parallel}{(\varepsilon_\perp - \varepsilon_\parallel)\sin^2\theta}\left(\frac{\varepsilon_\perp\sin^2\theta + \varepsilon_\parallel\cos^2\theta}{\varepsilon_\perp\varepsilon_\parallel}\right) \\ &\times\frac{c}{\omega}\frac{\mathrm{d}\phi}{\mathrm{d}z}\boldsymbol{r} - 2\mathrm{i}\,\varepsilon_\perp^{-1/2}\cot\theta\frac{c}{\omega}\frac{\mathrm{d}\phi}{\mathrm{d}z}\boldsymbol{k}\Bigg]\exp\mathrm{i}\left(\frac{\omega}{c}\int\varepsilon_\perp^{1/2}\,\mathrm{d}z - \omega t\right).\end{aligned} \tag{21}$$

The magnetic field $\boldsymbol{B}$ is known as $(-\mathrm{i}/\omega)\boldsymbol{\nabla}\times\boldsymbol{E}$. Here only $\boldsymbol{\nabla}_z$ contributes but, since so many of the factors are z-dependent, it would be complicated to set down specific formulae. In this approximation all the components that were absent in zero order now appear, proportional to $\mathrm{d}\phi/\mathrm{d}z$ and $\pi/2$ out of phase.

Now it is seen that this geometrical optics or WKB method also requires that the pitch be large compared with the wavelength. The series solution, equation (6), will only be sensible if the $E^{(1)}$-terms are small compared with the $E^{(0)}$-terms. The estimate of the ratio of the sizes of these terms in (20) and (21) is $(c/\omega)\,\mathrm{d}\phi/\mathrm{d}z$ so that the requirement is

$$(c/\omega)\,\mathrm{d}\phi/\mathrm{d}z < 1. \tag{22}$$

The left-hand side here is the ratio of the wavelength to the pitch.

6. Conclusions

The main results are formulae (18) and (19) for the electric and magnetic fields, in the first approximation, for the two modes of propagation. It is easy to visualise the result. In mode A the electric field has a longitudinal component; otherwise all the fields are transverse. The transverse parts of the fields are either parallel or perpendicular to the transverse component of the director and these polarisation directions change from layer to layer, tracking the director.

It is expected that this WKB series will have rapid convergence, since this is the case in the ordinary WKB treatment. Some quantitative remarks about this question can be made. The process gives a series solution to the coupled system (equations (4)). The usefulness of the series may be judged by comparing the different terms in the series, as in (15). Consider solution A and the terms proportional to m_A in $E^{(0)} + E^{(1)}$. After a lengthy calculation one finds

$$\overline{C}_A p_A^{-1/2} \boldsymbol{m}_A + \alpha \boldsymbol{m}_A \simeq \overline{C}_A \boldsymbol{m}_A \exp \mathrm{i} \int p_A \, \mathrm{d}z$$

$$\times \left[1 - \frac{\mathrm{i}}{4\pi} \frac{\mathrm{d}\lambda_A}{\mathrm{d}z} + \frac{1}{16\pi^2} \lambda_A \frac{\mathrm{d}}{\mathrm{d}z} \frac{\mathrm{d}\lambda_A}{\mathrm{d}z} - \frac{1}{32\pi^2} \left(\frac{\mathrm{d}\lambda_A}{\mathrm{d}z} \right)^2 - \frac{1}{2p_A^2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}z} \right)^2 + \frac{2}{p_A^2 - p_B^2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}z} \right)^2 \right]. \tag{23}$$

Here α was determined by requiring the equations for $E^{(2)}$ to be solvable. The $\mathrm{d}\lambda_A/\mathrm{d}z$ term in the integrand comes from $p_A^{-1/2} \boldsymbol{m}_A$ as in (15), and the last four terms in the integrand come from $\alpha \boldsymbol{m}_A$. In view of this result it is understood that first-order effects are marked by $\mathrm{d}\lambda/\mathrm{d}z$ or $(c/\omega)(\mathrm{d}\phi/\mathrm{d}z)$ factors and that second-order effects have $(\mathrm{d}\lambda/\mathrm{d}z)^2$, $(c/\omega)^2(\mathrm{d}\phi/\mathrm{d}z)^2$, or $\lambda(\mathrm{d}/\mathrm{d}z)(\mathrm{d}\lambda/\mathrm{d}z)$ factors.

A problem for the future is to analyse this same system for off-axis propagation. A preliminary investigation suggests that the same method will apply for paraxial waves, although the calculations will be considerably more complicated at every stage.

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